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Moduli spaces with external fields

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Abstract

We consider the geometric structures on the moduli space of static finite energy solutions to the $2+1$-dimensional unitary chiral model with the Wess–Zummino–Witten (WZW) term. It is shown that the magnetic field induced by the WZW term vanishes when restricted to the moduli spaces constructed from the Grassmannian embeddings, so that the slowly moving solitons can in some cases be approximated by a geodesic motion on a space of rational maps from $\mathbb{CP}^1$ to the Grassmannian.

Keywords: Moduli space dynamics; Integrable chiral model; Torsion

1. Introduction

Let us consider a fairly general framework for field theory. The *space time* (M, η) is a $(D+1)$-dimensional manifold with a Lorentzian metric η, and the *target space* (Y, h_Y) is a k-dimensional manifold with a (pseudo-) Riemannian metric h_Y. The dynamics of the theory depends on a choice of the action, which is a functional on the space of maps $\text{Map}(M, Y)$. In the *canonical* approach one sets $M = \Sigma \times \mathbb{R}$, and regards the field equations as the infinite dimensional dynamical system on the space $\mathcal{M}$ of maps $J : \Sigma \longrightarrow Y$, where the initial data set Σ is the D-dimensional manifold with a Riemannian metric induced by η. The details depend on the model, but generally one aims to formulate the evolution equations as the geodesic motion (possibly with a potential) on $\mathcal{M}$. The L^2 metric on $\mathcal{M}$ is induced by the target space metric h_Y in the following way. For a given map J we identify $T_J\mathcal{M}$ with the space of maps $X : \Sigma \longrightarrow TY$ such that $\pi \circ X = J$, where $\pi : TY \to Y$ is a natural projection, and set

$$|X|^2 = \int_\Sigma h_Y(X(p), X(p))\mathrm{d}p, \quad X \in T_J\mathcal{M}, \tag{1.1}$$

where $\mathrm{d}p$ is some measure on Σ.

In the case of gravity (where the overall structure differs slightly from the one described above) this leads to the exact procedure realising the Einstein equation as a dynamical system, where the metric on $\mathcal{M}$ is the celebrated deWitt metric, and the potential is given by the scalar curvature of the Riemannian metric on the initial data set Σ. To make it all work one needs to factor out $\mathcal{M}$ by the action of the group of diffeomorphisms of Σ, and ensure that the initial data satisfies the constraint equations [6].

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In gauge theory, where one considers the quotient of $\mathcal{M}$ by the infinite dimensional group of gauge transformations, the metric (1.1) coincides with the inner product induced by the kinetic term in the Lagrangian on M. Here the emphasis has been on the approximate techniques. In the moduli space approximation [9] the dynamics is restricted to a finite dimensional submanifold of $\mathcal{M}$. This submanifold consists of appropriately chosen static finite energy solutions to the full field equations. If the 'initial position' is given by a static solution which minimises the potential energy of the field configuration and initial kinetic energy is small, then the trajectory in $\mathcal{M}$ can be expected to stay close to a geodesic in the manifold of static finite energy solutions. This 'follows from' the total energy conservation. In some cases the whole procedure can be made rigorous [15]. See [10] for a review of the geodesic method.

In other cases (including various string theories in 10 dimensions and 11-dimensional super-gravity), the target space Y admits a rich structure consisting of more than just a (pseudo-)Riemannian metric. In particular any differential $(D+r)$-form on Y induces an r-form on $\mathcal{M}$ in a way which does not depend on h_Y. If this differential form appears in the Lagrangian it will give rise to an external (magnetic-like) field on the moduli space. This can lead to interesting physical consequences. If the topology of $\mathcal{M}$ is non-trivial the Aharonov–Bohm effect may take place on the space of solutions even if the magnetic field vanishes [23].

In this paper we shall give a detailed analysis of one example where an external field arise on the moduli space. We shall take the space–time M to be $\mathbb{R}^{2,1}$, and the target space to be a unitary Lie group with its natural trace form metric. Any Lie group admits a connection which parallel propagates left-invariant vector fields. This connection is flat, but necessarily has torsion. Using this connection with torsion in the chiral model Lagrangian modifies the equations of motion, and surprisingly makes them integrable [19]. This modification can also be interpreted in terms of the WZW term in the chiral model action. In the rest of this section we shall introduce this modified chiral model, originally due to Ward [18]. We shall also review its static solutions given in terms of the Grassmannian embeddings $\mathbb{CP}^n \longrightarrow U(n+1)$. In Section 2 we shall construct a metric and a magnetic potential on the moduli space of static solutions. The corresponding magnetic field will be shown to vanish, but the flat magnetic connection can still be interesting, since the moduli space (which in our case consists of based rational maps $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^n$, where the 2-sphere $\Sigma = S^2 = \mathbb{CP}^1$ is the initial data set compactified by the boundary conditions) is not simply connected. In Section 3 we shall show that the magnetic 1-form can be obtained canonically from a pull-back of a certain 1-form from $\mathbb{CP}^n$. In the Appendix we discuss the Noether currents arising from the WZW Lagrangian. Some of the results presented in this paper appeared in the MSc Thesis of the second author.

1.1. Modified chiral model

Consider a smooth map $J : \mathbb{R}^{2,1} \longrightarrow U(n+1)$. The integrable chiral model is defined by equation

$$\begin{aligned} &\left(\eta^{\mu\nu} - V_\alpha \epsilon^{\alpha\mu\nu}\right)(J^{-1}J_\mu)_\nu = 0, \\ &\eta = \mathrm{diag}(-1,1,1), \qquad V_\alpha = (0,1,0), \qquad \epsilon^{012} = 1, \end{aligned} \tag{1.2}$$

where Greek letters denote three-dimensional space–time indices taking values $0,1,2 \equiv t,x,y$. The abbreviated notation of differentiation $J_\mu \equiv \partial_\mu J$ and the summation convention is going to be used in the article. A choice of the unit space-like vector $V = \partial/\partial x$ breaks the Lorentz invariance down to $SO(1,1)$, but ensures the integrability of (1.2).

The Lagrangian formulation of (1.2) contains the Wess–Zumino–Witten (WZW) term [2,7]. This involves an extended field $\hat{J}$ defined in the interior of a cylinder which has the space–time as one of its boundary components

$$\hat{J} : \mathbb{R}^{2+1} \times [0,1] \longrightarrow U(n+1)$$

such that $\hat{J}(x^\mu, 0)$ is a constant group element, which we take to be the identity $\mathbf{1} \in U(n+1)$, and $\hat{J}(x^\mu, 1) = J(x^\mu)$.

The Eq. (1.2) can be derived as a stationary condition for the action functional

$$\begin{aligned} S &= S_C + S_M, \\ S_C &= -\frac{1}{2}\int_{[t_1,t_2]\times\mathbb{R}^2} \mathrm{Tr}\,(\mathbb{J} \wedge \star\mathbb{J}), \\ S_M &= \frac{1}{3}\int_{[t_1,t_2]\times\mathbb{R}^2\times[0,1]} \mathrm{Tr}\left(\hat{\mathbb{J}} \wedge \hat{\mathbb{J}} \wedge \hat{\mathbb{J}} \wedge \mathbb{V}\right), \end{aligned} \tag{1.3}$$

where J should be treated as a field. Here $\star$ is a Hodge star of $\eta_{\mu\nu}$ and

$$\mathbb{J} = J^{-1} J_\mu \mathrm{d}x^\mu, \qquad \hat{\mathbb{J}} = \hat{J}^{-1} \hat{J}_p \mathrm{d}x^p, \quad p = 0, 1, 2, 3 \equiv t, x, y, \rho$$

are $\mathfrak{u}(n+1)$-valued 1-forms on $\mathbb{R}^{2+1}$ and $\mathbb{R}^{2+1} \times [0, 1]$ respectively and $\mathbb{V} = \mathbf{1}\, \mathrm{d}x$ is a constant 1-form on $\mathbb{R}^{2+1} \times [0, 1]$. We make an assumption that the extension $\hat{J}$ is of the form

$$\hat{J}(x^\mu, \rho) = \mathcal{F}\left(J(x^\mu), \rho\right) \tag{1.4}$$

for some smooth function $\mathcal{F} : U(n+1) \times [0, 1] \longrightarrow U(n+1)$. The WZW term S_M in the action is topological in the sense that its integrand does not depend on the metric on $\mathbb{R}^{2,1}$.

Following [22] we can obtain a more geometric picture by regarding the domain of $\hat{J}$ as $B \times \mathbb{R}$, where B is a ball in $\mathbb{R}^3$ with the boundary $\partial B = S^2$ regarded as a compactified space, and rewriting S_M as

$$S_M = \int_{[t_1,t_2]\times B} \hat{J}^*(T) \wedge V, \quad V = \mathrm{d}x.$$

Here T is the preferred 3-form [2] on $U(n+1)$ in the third cohomology group given by $T = \mathrm{Tr}[(\phi^{-1}\mathrm{d}\phi)^3]$ for $\phi \in U(n+1)$. This 3-form coincides with torsion of a flat connection ∇ on $U(n+1)$ which parallel propagates left-invariant vector fields, i.e.

$$T(X, Y, Z) = g\left(\nabla_X Y - \nabla_Y X - [X, Y], Z\right),$$

where $g = -\mathrm{Tr}(\phi^{-1}\mathrm{d}\phi\ \phi^{-1}\mathrm{d}\phi)$ is the metric on $U(n+1)$ given in terms of the Maurer–Cartan 1-form (this definition makes sense for any matrix Lie group).

The torsion 3-form T can be pulled back to B. It is closed, so $T = \mathrm{d}\lambda$, where λ is a 2-form on G which can be defined only locally. The Stokes theorem now yields

$$\begin{aligned} S_M &= \int_{[t_1,t_2]\times B} \mathrm{d}(\hat{J}^*(\lambda) \wedge V) \\ &= \frac{1}{2} \int_{S^2\times[t_1,t_2]} (\varepsilon^{\mu\nu\alpha} V_\alpha) \lambda_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j \mathrm{d}x\mathrm{d}y\mathrm{d}t, \end{aligned}$$

where $\phi^i = \phi^i(x^\mu)$ are local coordinates on the group (e.g. the components of the matrix J). In the above derivation we have neglected the boundary component $(t_1 \times B) \cup (t_2 \times B)$, as variations of the corresponding integrals vanish identically.

Time translational invariance of S gives rise to the conservation of the energy functional which appears to be the same as for the ordinary chiral model[1]

$$\begin{aligned} E &= T + E_p, \\ T &= -\frac{1}{2} \int_{\mathbb{R}^2} \mathrm{Tr}\left((J^{-1}J_t)^2\right) \mathrm{d}x\mathrm{d}y, \\ E_p &= -\frac{1}{2} \int_{\mathbb{R}^2} \mathrm{Tr}\left((J^{-1}J_x)^2 + (J^{-1}J_y)^2\right) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{1.5}$$

Finiteness of energy can be ensured [18] by imposing the boundary condition on J

$$J(t, r, \theta) = J_0 + r^{-1} J_1(\theta) + O(r^{-2}), \qquad x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}, \tag{1.6}$$

where J_0 is a constant matrix [18] and the whole dependence on t lies in $O(r^{-2})$.

1.2. Grassmannian models

A Grassmannian model in $2 + 1$ dimensions is defined by the equation

$$[\partial_\mu \partial^\mu P, P] = 0, \tag{1.7}$$

[1] In the Appendix we shall construct the corresponding momenta in x and y directions.

where P is a map from $\mathbb{R}^{2+1}$ into the Grassmannian manifold $\mathrm{Gr}(m, n+1)$ of complex m-dimensional linear subspaces in $\mathbb{C}^{n+1}$. We shall think of P as a complex Hermitian matrix of rank m such that $P^2 = P$.

The field equation (1.7) can be derived from the action

$$S = 2 \int_{[t_1,t_2]\times\mathbb{R}^2} \mathrm{Tr}\,(\mathbb{P} \wedge \star\mathbb{P})\,, \tag{1.8}$$

where $\mathbb{P} = P_\mu \mathrm{d}x^\mu$. For $m = 1$ the Grassmannian models reduce to the $\mathbb{CP}^n$ models.

1.3. Moduli space approximation

All finite energy static solutions to (1.2) can be factorised in terms of maps $P_{(\alpha)}$ of $\mathbb{R}^2$ into Grassmannian manifolds [16,24]

$$J = K(\mathbf{1} - 2P_{(1)})(\mathbf{1} - 2P_{(2)}) \cdots (\mathbf{1} - 2P_{(\mathcal{N})}), \tag{1.9}$$

where K is a constant unitary matrix, $P_{(\alpha)}$ satisfy some first order PDEs, and $\mathcal{N} \leq n$ is the so called uniton number. It can happen that all uniton factors can be shrunk into the form

$$J = K(\mathbf{1} - 2P), \tag{1.10}$$

where P maps $\mathbb{R}^2$ into some Grassmannian manifold but does not necessarily satisfy the first order PDEs involved in the definition of unitons. It can be easily checked that (1.10) is a solution to chiral model if and only if P is a solution to Grassmannian model. We will call such solutions *Grassmannian embeddings*. Note that they can represent one-uniton as well as particular multi-uniton solutions. One-uniton solutions correspond to P being (anti)instanton solution, which at the level of the Grassmannian model minimises the value of energy in its topological sector. For such solutions the energy is proportional to the topological charge of the Grassmannian projector (given by the formula (3.2) in Section 3). This is also true for the potential energy of the chiral field J, defined in (1.5), since in the case of (1.10) it is equal to the energy of P.

Integrability of the model enables a construction of time dependent solutions by twistor and inverse scattering methods [18,20]. Approximate solutions corresponding to low energy exact solutions can also be sought by a modification of Manton's geodesic approximation. The modification relies on taking into account a background magnetic field in the moduli space of static solutions induced by WZW term in (1.3), and has been discussed in [4] for the $SU(2)$ models. In this reference the moduli space has been constructed from static solutions of the model obtained by embedding the instanton solutions of the $\mathbb{CP}^1$ model (which together with an analogous procedure of embedding anti-instantons gives all static solutions in the $SU(2)$ case). It has been demonstrated that the magnetic field vanishes and so the integrable $SU(2)$ chiral model appears to be equivalent to the usual $SU(2)$ chiral model at the level of the approximation. The proof given in [4] relied on the fact that $SU(2)$ is three dimensional, and it remained uncertain how the magnetic field behaves for higher dimensions of the target manifold. The main result of this paper is to clarify this and to show that the magnetic field vanishes on moduli spaces constructed from the Grassmannian embeddings into $U(n+1)$ models for arbitrary n. The static Grassmannian solutions will not be required to be instantons or anti-instantons in our proof.

2. The metric and the magnetic field on the moduli space

The boundary conditions (1.6) imply that the finite energy static solutions to (1.2) are maps from S^2 (conformal compactification of $\mathbb{R}^2$) into $U(n+1)$. In the moduli space approximation we choose a class of such solutions which are homotopic as maps of S^2 into $U(n+1)$ and all have the same value of potential energy. Ideally every such map ought to provide minimum of the potential energy. This is the case on the level of the Grassmannian models for constructions which involve (anti)instanton solutions. For chiral models one can show that all finite energy static solutions are saddle points of the potential energy functional [11]. This raises a question about stability of the approximate solutions.

For a given value of the topological charge, all solutions in the class can be described by finite set of parameters, which in the case of instantons are positions of zeros and poles of holomorphic functions. To ensure finite values of kinetic energy we need to impose the *base condition* on the solutions by fixing their value at spatial infinity. Then the parameters, if chosen appropriately, may define a map on the resulting moduli space. Next we allow the parameters to depend on time and so time dependent approximate solutions correspond to paths in the moduli space. Let us denote the solutions contributing to the moduli space by $J(\gamma; x, y)$, where γ denote real parameters. Approximate time dependent solutions are then of the form $J(\gamma(t); x, y)$ and time differentiation gives

$$J_t = J_j \dot{\gamma}^j, \quad j = 1, \ldots, \dim \mathcal{M}. \tag{2.1}$$

The dynamics is governed by the action obtained as a restriction of (1.3) to the moduli space

$$S_{\mathcal{M}} = \int_{t_1}^{t_2} \left(\frac{1}{2} h_{jk} \dot{\gamma}^j \dot{\gamma}^k + A_j \dot{\gamma}^j \right) \mathrm{d}t. \tag{2.2}$$

The metric term can be obtained from kinetic energy form (1.5) by use of (2.1)

$$T = \frac{1}{2} h_{jk} \dot{\gamma}^j \dot{\gamma}^k, \qquad h_{jk} = -\int \mathrm{Tr}\left(J^{-1} J_j J^{-1} J_k\right) \mathrm{d}x\mathrm{d}y, \tag{2.3}$$

and the magnetic term can similarly be obtained from the WZW term, which can be rewritten by cyclic property of the trace as

$$\begin{aligned} S_M &= \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_0^1 \mathrm{Tr}([\hat{J}^{-1}\hat{J}_t, \hat{J}^{-1}\hat{J}_y]\hat{J}^{-1}\hat{J}_\rho) \mathrm{d}\rho \mathrm{d}x \mathrm{d}y \mathrm{d}t \\ &= \int_{t_1}^{t_2} A_j \dot{\gamma}^j \mathrm{d}t, \end{aligned}$$

where

$$A_j = \int_{\mathbb{R}^2} \int_0^1 \mathrm{Tr}([\hat{J}^{-1}\hat{J}_j, \hat{J}^{-1}\hat{J}_y]\hat{J}^{-1}\hat{J}_\rho) \mathrm{d}\rho \mathrm{d}x \mathrm{d}y. \tag{2.4}$$

Then $A = A_j d\gamma^j$ is the *magnetic* 1*-form* on the moduli space. We shall now prove the following

Theorem 2.1. *The magnetic field (2.5) vanishes on moduli spaces constructed from embeddings (1.10) of Grassmannian solutions.*

Proof. The essence of WZW term is that its variation does not depend on the particular choice of the extension $\hat{J}$. We consider the variations restricted to the moduli space $\delta J = J_i \delta\gamma^i$, and find

$$\begin{aligned} \delta S_M &= \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \mathrm{Tr}\left(J^{-1} J_y [J^{-1} \delta J, J^{-1} J_t]\right) \mathrm{d}x \mathrm{d}y \mathrm{d}t \\ &= -\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \mathrm{Tr}\left(J^{-1} J_y [J^{-1} J_i, J^{-1} J_j]\right) \mathrm{d}x \mathrm{d}y \dot{\gamma}^j \delta\gamma^i \mathrm{d}t. \end{aligned}$$

Comparing this expression with the variation of (2.4)

$$\delta S_M = \int_{t_1}^{t_2} F_{ij} \dot{\gamma}^j \delta\gamma^i \mathrm{d}t, \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

gives

$$F_{ij} = -\int_{\mathbb{R}^2} \mathrm{Tr}\left(J^{-1} J_y [J^{-1} J_i, J^{-1} J_j]\right) \mathrm{d}x \mathrm{d}y, \tag{2.5}$$

where $F = \frac{1}{2} F_{ij}(\gamma) \mathrm{d}\gamma^i \wedge \mathrm{d}\gamma^j$ is the *the magnetic field*. We can see, that although the magnetic 1-form A in general depends on the choice of the extension $\hat{J}$, its exterior derivative F does not. Changing the extension merely corresponds to a gauge transformation of A.

Note that the potential energy term has not been included in the effective action (2.2). The potential is proportional to the topological charge (3.2), and does not contribute to the effective equations of motion.

Let now us consider a Grassmannian projector P depending smoothly on some set of variables, which we shall denote by a, b, c. From idempotency and the Leibniz rule we deduce

$$P_a = P_a P + P P_a,$$

and

$$\begin{aligned} P_a P_b P_c &= P P_a P_b P_c + P_a P P_b P_c = P P_a P_b P_c + P_a P_b P_c - P_a P_b P P_c \\ &= P P_a P_b P_c + P_a P_b P_c - P_a P_b P_c + P_a P_b P_c P = P P_a P_b P_c + P_a P_b P_c P. \end{aligned}$$

Taking the trace of the above expression gives

$$\mathrm{Tr}(P_a P_b P_c) = \mathrm{Tr}(2 P P_a P_b P_c). \tag{2.6}$$

If J is given by (1.10) then

$$\mathrm{Tr}\left(J^{-1} J_y [J^{-1} J_i, J^{-1} J_j]\right) \sim \mathrm{Tr}\left((1 - 2P) P_y [P_i, P_j]\right), \tag{2.7}$$

where we have assumed that K does not depend on parameters γ on the moduli space to ensure the finiteness of the kinetic energy. The RHS of (2.7) vanishes because of (2.6), which in turn implies the vanishing of the magnetic field (2.5). $\square$

3. Canonical structures on the moduli space

The $\mathbb{CP}^n$ models have been discussed in detail within the moduli space approach [17,14]. It is convenient to choose a map and perform calculation in a local framework. We can represent complex directions in $\mathbb{C}^{n+1}$, which are the elements of $\mathbb{CP}^n$, by vectors in $\mathbb{C}^{n+1}$ with their first component fixed to 1. Then the map f defined by

$$\mathbb{CP}^n \ni (1, f^1, \ldots, f^n) \longrightarrow (f^1, \ldots, f^n) \in \mathbb{C}^n \tag{3.1}$$

belongs to the maximal holomorphic atlas of $\mathbb{CP}^n$. The results do not depend on the choice of this map. The topological charge for the $\mathbb{CP}^n$ models is

$$Q = -\mathrm{i} \int_{\mathbb{R}^2} \mathrm{Tr}(P[P_x, P_y]) \mathrm{d}x \mathrm{d}y = -\frac{1}{4} \int_{\mathbb{R}^2} P^* \Phi, \tag{3.2}$$

where

$$\Phi = -4\mathrm{i}\partial\bar{\partial} \ln\left(1 + \sum_{l=1}^{n} |f^l|^2\right) = -4\mathrm{i} \frac{\delta^{jk}\left(1 + \sum_{l=1}^{n} |f^l|^2\right) - f^j \bar{f}^k}{\left(1 + \sum_{l=1}^{n} |f^l|^2\right)^2} \mathrm{d}f^k \wedge \mathrm{d}\bar{f}^j \tag{3.3}$$

is the Kähler form of the Fubini–Study metric on $\mathbb{CP}^n$ and $P^*\Phi$ denotes its pull-back. The first expression for Q given in (3.2) is often more convenient for calculations, while the second clarifies the topological character. The matrix P is given in terms of f^l by

$$P = \frac{W \otimes W^\dagger}{W^\dagger W}, \quad W = \begin{pmatrix} 1 \\ f^1 \\ \vdots \\ f^n \end{pmatrix}. \tag{3.4}$$

The equality (3.2) is proved by establishing that in the chosen map both expressions give

$$-\mathrm{i} \int_{\mathbb{R}^2} \sum_{k,j=1}^{n} \frac{\delta^{kj}\left(1 + \sum_{l=1}^{n} |f^l|^2\right) - \bar{f}^k f^j}{\left(1 + \sum_{l=1}^{n} |f^l|^2\right)^2} \frac{\partial(f^k, \bar{f}^j)}{\partial(x, y)} \mathrm{d}x \mathrm{d}y. \tag{3.5}$$

The most natural choice of the family of static solutions for the purpose of construction of the moduli space is to consider solutions which minimise the energy for a given value of topological charge. These instanton (or anti-instanton) solutions correspond to $f^l, l = 1 \ldots n$ being rational holomorphic (respectively antiholomorphic) functions of the complex variable $z = x + \mathrm{i}y$. Let us concentrate on instantons, in which case [24]

$$Q = 2\pi N, \tag{3.6}$$

where $N = \max_l(k_{\mathrm{alg}} f^l)$ is an integer. Here $k_{\mathrm{alg}} f^l$ is the algebraic degree of the rational function f^l. Note that (3.6) holds for any smooth map $P : S^2 \longrightarrow \mathbb{CP}^n$ with N being the homotopy class under the standard isomorphism $\pi_2(\mathbb{CP}^n) = \mathbb{Z}$. To see it (e.g. [1]) consider the homology group $H_2(\mathbb{CP}^n)$. This is isomorphic to $\mathbb{Z}$. If $P : S^2 \longrightarrow \mathbb{CP}^n$ is a map from the compactified space to $\mathbb{CP}^n$, representing a homology class $P_*[S^2]$, we obtain the corresponding integer by evaluating $P_*[S^2]$ on a standard generator for $H^2(\mathbb{CP}^n)$ represented by the Kahler form Φ. In terms of differential forms, evaluating a cohomology class on a homology class just means integrating, so the evaluation of $P_*[S^2]$ on Φ is given by the RHS of (3.2). Now consider the Hurewicz homomorphism from $\pi_2(\mathbb{CP}^n)$ to $H_2(\mathbb{CP}^n)$ sending the homotopy class of $P : S^2 \longrightarrow \mathbb{CP}^n$ to $P_*[S^2]$, where $[S^2] \in H_2(S^2)$ is the fundamental class. The projective space $\mathbb{CP}^n$ is simply connected, so this is an isomorphism $\pi_2(\mathbb{CP}^n) = H_2(\mathbb{CP}^n) = \mathbb{Z}$.

For a given N, the finiteness of the energy requires the base condition to be imposed. We therefore fix the limit of each f^l at the spatial infinity. Let us choose this limit to be equal to one for all functions f^l. Then they are of the form

$$f^l = \frac{p_l(z)}{q_l(z)} = \frac{(z - q^{l,1}) \ldots (z - q^{l,N})}{(z - q^{l,N+1}) \ldots (z - q^{l,2N})}, \quad l = 1, \ldots, n, \tag{3.7}$$

and complex numbers q are holomorphic coordinates on a finite dimensional moduli space $\mathcal{M}_N \subset \mathcal{M}$. We can define the metric as a restriction of kinetic energy form to $\mathcal{M}_N$ (like in (2.3)). Its completeness is obviously equivalent to the requirement that the kinetic energy is finite along all curves in $\mathcal{M}_N$. Although the base condition was necessary to ensure finite kinetic energies, it appears not to be sufficient as the metric is complete only on leaves of appropriate foliation of $\mathcal{M}_N$ [17,14] and we need to restrict the dynamics to these leaves. These restrictions are assumed to hold in the rest of the paper and we will often use the symbol $\mathcal{M}_N$ to denote some particular leaf.

The metric described above, which can be obtained explicitly from (2.3) by use of (1.10), is Kähler with respect to the natural complex structure induced by map (3.7), with the Kähler potential

$$\Omega = 8 \int_{\mathbb{R}^2} \ln \sum_{l=1}^{n} (|p_l|^2 + |q_l|^2) \mathrm{d}x\mathrm{d}y. \tag{3.8}$$

As noted in [13] this metric can also be defined canonically. Let γ denote the set of real parameters of all rational functions (3.7), which provide real coordinates on $\mathcal{M}_N$. For example one may consider $\gamma = ((q + \overline{q})/2, (q - \overline{q})/(2i))$. In the following, γ will also denote a point in $\mathcal{M}_N$. Let $\{P(\gamma; \cdot) : \mathbb{R}^2 \longrightarrow \mathbb{CP}^n, \gamma \in \mathcal{M}_N\}$ be instanton solutions of the model. We can define the maps

$$F : \mathcal{M}_N \times \mathbb{R}^2 \longrightarrow \mathbb{CP}^n, \qquad F(\gamma, p) := P(\gamma; p), \tag{3.9}$$

$$\begin{aligned} F_p &:= F(\cdot, p) : \mathcal{M}_N \longrightarrow \mathbb{CP}^n \\ F_\gamma &:= F(\gamma, \cdot) : \mathbb{R}^2 \longrightarrow \mathbb{CP}^n. \end{aligned} \tag{3.10}$$

For each smooth vector field on $\mathcal{M}_N$

$$X : \mathcal{M}_N \longrightarrow T\mathcal{M}_N, \quad X \in T_\gamma \mathcal{M}_N$$

we can now define the metric h canonically by

$$h(X, X) = \int_{\mathbb{R}^2} \hat{h} \left(F_{p*}X, F_{p*}X \right) \mathrm{d}x\mathrm{d}y, \tag{3.11}$$

where $F_{p*}X$ denotes the push-forward of a vector field, $\hat{h}$ is a Fubini–Study metric on $\mathbb{CP}^n$ and integration is performed with respect to $p \equiv (x, y)$.

Let us now observe that all the results discussed here for $\mathbb{CP}^n$ models can be extended to the chiral models. To see it consider the moduli space constructed from $\mathbb{CP}^n$ embeddings (1.10), where for convenience we set $K = -i\mathbf{1}$:

$$J = -\mathrm{i}(\mathbf{1} - 2P). \tag{3.12}$$

Since the kinetic energy for chiral models rewritten in terms of P is precisely the one for $\mathbb{CP}^n$ models, the Kähler structure is also the same. Thus the moduli space $\mathcal{M}_N$ can be considered as an arena for slowly moving $\mathbb{CP}^n$ Skyrmions, as well as the low energy solutions to the $U(n+1)$ Ward model. The magnetic 1-form (2.4) is an interesting object in spite of the fact that it does not influence the motion. In Theorem 3.1 we shall show how it arises canonically on the moduli space. Let us first make some comments about the extensions of J used in the variational principle.

In general any J can be extended, as the obstruction group $\pi_2(U(n+1))$ vanishes. In the case of soliton solutions to (1.2) we can be more explicit. It has been shown in [3] that all solitons factorise as $J = \prod_\alpha M_\alpha$ into a finite number of the time dependent unitons of the form $M_\alpha = \mathbf{1} - (1 - \mathrm{e}^{2\mathrm{i}\phi_\alpha})P_\alpha$, where $P_\alpha = P_\alpha(x, y, t)$ are Hermitian projectors, and the real constants ϕ_α are the phases of the poles on the spectral plane. Any of these projectors can be extended by

$$M_\alpha \longrightarrow \hat{M}_\alpha = \mathbf{1} - (1 - \mathrm{e}^{2\mathrm{i}\rho\phi_\alpha})P_\alpha, \tag{3.13}$$

thus giving the extension $\hat{J} = \prod_\alpha \hat{M}_\alpha$. In the next theorem we shall use an extension

$$\hat{J}(t, x, y, \rho) = \cos g(\rho)\mathbf{1} + \sin g(\rho) J(t, x, y). \tag{3.14}$$

This extension corresponds to $\mathcal{F}(J, x, y, \rho) = \cos g(\rho)\mathbf{1} + \sin g(\rho) J$, however the domain of $\mathcal{F}$ should be restricted from to $\mathcal{U} \times \mathbb{R}^2 \times [0, 1]$, where

$$\mathcal{U} := \{J = -\mathrm{i}(\mathbf{1} - 2P) : P \in \mathbb{CP}^n\}.$$

Such restriction is allowed, since all mappings J within the moduli space take values in $\mathcal{U} \subset U(n+1)$. In the case of static solutions (3.12) the extensions (3.13) and (3.14) differ only by an overall factor depending on ρ, which does not contribute to the magnetic 1-form.

Theorem 3.1. *The magnetic* 1*-form* (2.4) *induced by* WZW *term for the extension* (3.14) *coincides with the canonical 1-form on $\mathcal{M}_N$ defined by*

$$A_{\mathrm{kan}}(X) = \frac{\pi}{2}\int_{\mathbb{R}^2} \hat{h}\left(F_{\gamma *}V, F_{p*}X\right)\mathrm{d}x\mathrm{d}y \tag{3.15}$$

where $V = \partial/\partial x$ is the unit vector defining the Ward equation (1.2).

Proof. Let us compare components of both 1-forms in the map γ. From (3.15) we find

$$\begin{aligned}(A_{\mathrm{kan}})_j &= A_{\mathrm{kan}}\left(\frac{\partial}{\partial \gamma^j}\right) = \frac{\pi}{2}\int_{\mathbb{R}^2} \hat{h}\left(F_{\gamma *}V(p), F_{p*}\frac{\partial}{\partial \gamma^j}\right)\mathrm{d}x\mathrm{d}y \\ &= \frac{\pi}{2}\int_{\mathbb{R}^2} \Phi\left(\mathcal{J}F_{\gamma *}V(p), F_{p*}\frac{\partial}{\partial \gamma^j}\right)\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where $\mathcal{J}$ is the standard complex structure on $\mathbb{CP}^n$ given by $\mathcal{J}\frac{\partial}{\partial f^l} = \mathrm{i}\frac{\partial}{\partial f^l}$. Since

$$\mathcal{J}F_{\gamma *}V(p) = \mathrm{i}\frac{\partial f^k}{\partial z}\frac{\partial}{\partial f^k} - \mathrm{i}\frac{\partial \bar{f}^k}{\partial \bar{z}}\frac{\partial}{\partial \bar{f}^k}, \qquad F_{p*}\frac{\partial}{\partial \gamma^j} = \frac{\partial f^k}{\partial \gamma^j}\frac{\partial}{\partial f^k} + \frac{\partial \bar{f}^k}{\partial \gamma^j}\frac{\partial}{\partial \bar{f}^k},$$

we can use linearity and antisymmetry of the Fubini–Study Kähler form Φ given by (3.3) to obtain

$$(A_{\mathrm{kan}})_j = \frac{\pi}{2}\int_{\mathbb{R}^2} \mathrm{i}\left\{f_z^k \bar{f}_j^l \Phi\left(\frac{\partial}{\partial f^k}, \frac{\partial}{\partial \bar{f}^l}\right) - \bar{f}_{\bar{z}}^k f_j^l \Phi\left(\frac{\partial}{\partial \bar{f}^k}, \frac{\partial}{\partial f^l}\right)\right\}\mathrm{d}x\mathrm{d}y,$$

where the short notation $\partial f^k/\partial z = f_z^k,\ \partial f^k/\partial \gamma^j = f_j^k$ has been used. Since Φ is a fundamental form of a Hermitian metric, we have

$$\Phi\left(\frac{\partial}{\partial \bar{f}^k}, \frac{\partial}{\partial f^l}\right) = \overline{\Phi\left(\frac{\partial}{\partial f^k}, \frac{\partial}{\partial \bar{f}^l}\right)},$$

so

$$(A_{\text{kan}})_j = -\pi \int_{\mathbb{R}^2} \operatorname{Im}\left\{ f_z^k \bar{f}_j^l \Phi\left(\frac{\partial}{\partial f^k}, \frac{\partial}{\partial \bar{f}^l}\right)\right\} \mathrm{d}x\mathrm{d}y.$$

Finally we use (3.3) to obtain

$$(A_{\text{kan}})_j = \pi \int_{\mathbb{R}^2} \frac{4}{\left(1+\sum\limits_{r=1}^{n}|f^r|^2\right)^2} \operatorname{Re}\left\{\left(1+\sum_{r=1}^{n}|f^r|^2\right) f_z^l \bar{f}_j^l - f_z^k \bar{f}^k f^l \bar{f}_j^l\right\} \mathrm{d}x\mathrm{d}y. \tag{3.16}$$

Let us now consider the 1-form (2.4) induced by the WZW term. Substituting (3.14) into (2.4), performing ρ integration and using (3.12) yields

$$A_j = -2\pi \mathrm{i} \int_{\mathbb{R}^2} \operatorname{Tr}(P[P_j, P_y]).$$

To rewrite this expression in terms of rational functions f^k one needs to use (3.4). Then it is straightforward but laborious to obtain

$$A_j = 2\pi \mathrm{i} \int_{\mathbb{R}^2} \sum_{k,l=1}^{n} \frac{\delta^{kl}\left(1+\sum\limits_{r=1}^{n}|f^r|^2\right) - \bar{f}^k f^l}{\left(1+\sum\limits_{r=1}^{n}|f^r|^2\right)^2} \frac{\partial(f^k, \bar{f}^l)}{\partial(\gamma^j, y)} \mathrm{d}x\mathrm{d}y. \tag{3.17}$$

For holomorphic functions f^k we have

$$\frac{\partial(f^k, \bar{f}^l)}{\partial(\gamma^j, y)} = -\mathrm{i} f_j^k \bar{f}_z^l - \mathrm{i} \bar{f}_j^l f_z^k,$$
$$\delta^{kl} \frac{\partial(f^k, \bar{f}^l)}{\partial(\gamma^j, y)} = -2\mathrm{i} \operatorname{Re}(\bar{f}_z^k f_j^k), \qquad -\bar{f}^k f^l \frac{\partial(f^k, \bar{f}^l)}{\partial(\gamma^j, y)} = 2\mathrm{i} \operatorname{Re}(f^k \bar{f}_j^k f_z^l \bar{f}^l).$$

These formulae and (3.17) lead to

$$A_j = \int_{\mathbb{R}^2} \frac{4\pi}{\left(1+\sum\limits_{r=1}^{n}|f^r|^2\right)^2} \operatorname{Re}\left\{\left(1+\sum_{r=1}^{n}|f^r|^2\right) f_z^l \bar{f}_j^l - f_z^k \bar{f}^k f^l \bar{f}_j^l\right\} \mathrm{d}x\mathrm{d}y \tag{3.18}$$

which is the same as (3.16). □

Similarly, by this method we can easily prove that Ruback's metric (3.11) is equal to the metric (2.3), obtained as a reduction of kinetic energy form to $\mathcal{M}_N$. To see it write the components of (3.11)

$$\begin{aligned} h_{jk} &= h\left(\frac{\partial}{\partial \gamma^j}, \frac{\partial}{\partial \gamma^k}\right) = \int_{\mathbb{R}^2} \hat{h}\left(F_{p*}\frac{\partial}{\partial \gamma^j}, F_{p*}\frac{\partial}{\partial \gamma^k}\right) \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \hat{h}\left(\frac{\partial f^r}{\partial \gamma^j}\frac{\partial}{\partial f^r} + \frac{\partial \bar{f}^r}{\partial \gamma^j}\frac{\partial}{\partial \bar{f}^r}, \frac{\partial f^s}{\partial \gamma^k}\frac{\partial}{\partial f^s} + \frac{\partial \bar{f}^s}{\partial \gamma^k}\frac{\partial}{\partial \bar{f}^s}\right) \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \frac{8}{\left(1+\sum\limits_{l=1}^{n}|f^l|^2\right)^2} \operatorname{Re}\left\{\left(1+\sum_{l=1}^{n}|f^l|^2\right) f_j^r \bar{f}_k^r - f_j^s \bar{f}^s f^r \bar{f}_k^r\right\} \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.19}$$

On the other hand the substitution of (3.4) into (2.3) yields

$$
\begin{aligned}
2T &= \int_{\mathbb{R}^2} \frac{8}{(W^\dagger W)^2} \left\{ W^\dagger W W^\dagger{}_t W_t - W^\dagger W_t W^\dagger{}_t W \right\} \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2} \frac{8}{\left(1+\sum\limits_{l=1}^{n} |f^l|^2\right)^2} \left\{ \left(1+\sum_{l=1}^{n} |f^l|^2\right) f_t^r \bar{f}_t^r - f_t^s \bar{f}^s f^r \bar{f}_t^r \right\} \mathrm{d}x\mathrm{d}y. \qquad (3.20)
\end{aligned}
$$

4. Conclusions

Chiral models in $2+1$ dimensions can be made integrable by addition of a Wess–Zumino–Witten term to the standard action. This additional term gives rise to the magnetic 1-form on the moduli space of based rational maps $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^n$, i.e. the space of instanton solutions of the $\mathbb{CP}^n$ model, possibly embedded into chiral models. The magnetic 1-form depends on the choice of extension of the chiral field involved in the definition of the WZW term, but different extensions correspond to the $U(1)$ gauge transformations of the 1-form. The push-forward of the space-like unit vector appearing in (1.2) to the target space canonically defines a 1-form on such moduli space and we have shown that there exists a preferred gauge, which makes the WZW induced magnetic 1-form equal to the one obtained canonically.

The $U(1)$ connection defined by the magnetic 1-form is flat. This is the case not only for moduli spaces of instanton $\mathbb{CP}^n$ solutions, as the magnetic field vanishes on all moduli spaces constructed from Grassmannian embeddings. These results generalise the analysis of [4] which applies only to the target space $SU(2)$. A treatment of the moduli spaces of non-commutative solitons in the integrable $U(n+1)$ chiral model was recently given in [8], where even the Abelian case $n=0$ leads to a non-trivial structure.

In the case of $U(2)$ model there are no more possibilities, since here all static solutions are necessarily Grassmannian embeddings. It remains to be seen whether it is possible to construct the moduli spaces from static non-Grassmannian solutions of $U(n+1)$ model for $n>1$, such that the field would not vanish. For the Grassmannian embeddings the vanishing of the field is implied by vanishing of its density, the integrand of (2.5). It is possible to construct a moduli space for the $U(3)$ model such that this density does not vanish, however it seems to possess symmetries which ensure vanishing of the integral. This has been checked only for a few points in the moduli space, so the problem is open.

The moduli space approach to the ordinary $\mathbb{CP}^n$ model in $2+1$ dimensions does not approximate the true dynamics of the model. This has recently been shown by Rodnianski and Sterbenz [12] by a rigorous analysis of the (non-integrable) equations of motion. Rodnianski and Sterbenz have demonstrated that a class of solutions must blow up in finite time and a rate of this blow up is different than predicted by the geodesic approximation. The situation for the modified chiral model (1.2) is quite different, as there exist exact solutions which are regular for all times [18,21,3]. For some of these solutions the total (kinetic + potential) energy is quantised at the classical level by the elements of $\pi_3(U(n+1))$ [5], and thus for all t the total energy is equal to the potential energy of some static solution which in turn is equal to the degree (3.6) of some Grassmannian projector. Solutions to (1.2) obtained in the moduli space approximation presented in this paper have energies close to their potential energy as their kinetic energy is small. We should therefore expect that some of these approximate solutions arise from exact solutions by a limiting procedure.

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Appendix

The model (1.2) is translationally invariant and one expects to find the conserved momentum corresponding to the space translations. In [18] Ward has observed that the total energy and the y-momentum for (1.2) are the same as for the ordinary chiral model, but the x-momentum of the chiral model is not conserved by the time evolution (1.2) of the initial data. Here we shall revisit this problem and find the x-momentum using the WZW Lagrangian (1.3) written in terms of the torsion on $U(n+1)$. The Lagrangian density takes the form

$$
\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij}(\phi) + \frac{1}{2}V_\alpha\varepsilon^{\alpha\mu\nu}\lambda_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j,
$$

where g is the metric on the group, and λ is a local 2-form potential for the totally antisymmetric torsion [19]. The conserved Noether energy–momentum tensor is

$$T_{\mu\nu} = \eta_{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^j)}\partial_\nu \phi^j.$$

The energy corresponding to T_{00} is given by (1.5), and the momentum densities are

$$\mathcal{P}_y = T_{02} = -\mathrm{Tr}\left(J^{-1}J_t J^{-1}J_y\right),$$
$$\mathcal{P}_x = T_{01} = -\mathrm{Tr}\left(J^{-1}J_t J^{-1}J_x\right) - \lambda_{ij}\partial_x\phi^i\partial_y\phi^j. \tag{A.1}$$

The additional term in the conserved x-momentum $P_x = \int_{\mathbb{R}^2}\mathcal{P}_x \mathrm{d}x\mathrm{d}y$ does not depend on the choice of λ, since for a fixed t

$$\Theta := \int_{\mathbb{R}^2}\lambda_{ij}(\phi)\partial_x\phi^i\partial_y\phi^j \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} J^*\lambda. \tag{A.2}$$

This expression does not change under the transformation $\lambda \to \lambda + \mathrm{d}\beta$ because $\int_{\mathbb{R}^2}\mathrm{d}(J^*\beta) = 0$ as a consequence of the boundary condition (1.6). We can therefore choose the extension $\hat{J}$ given by (3.13) to find the additional term Θ using the identity

$$\int_{\mathbb{R}^2}\lambda_{ij}\partial_x\phi^i\partial_y\phi^j \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2}\int_0^1 \mathrm{Tr}\left(\hat{J}^{-1}\hat{J}_\rho\left[\hat{J}^{-1}\hat{J}_y, \hat{J}^{-1}\hat{J}_x\right]\right)\mathrm{d}\rho\mathrm{d}x\mathrm{d}y, \tag{A.3}$$

which follows from calculating $\mathcal{P}_x$ in terms of $\hat{J}$ directly from (1.3).

Consider the time dependent one-soliton solution [18]

$$J = \mathrm{i}\left(\mathbf{1} - \left(1 - \frac{\mu}{\bar{\mu}}\right)P\right).$$

Here $\mu \in \mathbb{C}/\mathbb{R}$ is a non-real constant, $P = W \otimes W^\dagger/\|W\|^2$ is the Grassmannian projection (3.4) and the components of $W : \mathbb{R}^{2,1} \to \mathbb{C}^{n+1}$ are holomorphic and rational in $\omega = x + \frac{\mu}{2}(t+y) + \frac{\mu^{-1}}{2}(t-y)$. In this case the additional term Θ is proportional to the topological charge (3.2), which is itself a constant of motion as the time evolution is continuous.

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